Math 255A Lecture 13 Notes

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1 Non-Solvability of Lewy's Operator

1.1 Continuity of bilinear forms

Here is a slight reformulation of a theorem we proved last lecture.

Theorem 1.1. *Let E be a locally convex space with the topology defined by countably many seminorms (not necessarily Hausdorff), F be a Fréchet space, and let G be locally convex space. Let $B : E \times F \to G$ be bilinear such that for all $x \in E$, $y \mapsto B(x, y)$ is continuous. If B is not continuous, then the set of all $y \in F$ such that $x \mapsto B(x, y)$ is continuous is a set of the first category.*

The proof is roughly the same, as well. We sketch it briefly.

Proof. Let $A_j = \{y \in F : B(x, y) \in \overline{U}\ \forall x \in V_j\}$, where U is a neighborhood of 0 in G and V_j form a fundamental system of neighborhoods of 0 in E. Then A_j is closed, convex, and symmetric. We claim that if $y \in F$ is such that $x \mapsto B(x, y)$ is continuous, then $y \in A_j$ for some j. If A_j has a nonempty interior for some j, then B is continuous. Thus if B is not continuous, the set $\{y \in F : x \mapsto B(x, y) \text{ continuous}\} \subseteq \bigcup_j A_j$ is of the first category. $\square$

1.2 Non-solvability of Lewy's operator

Theorem 1.2 (H. Lewy, 1957)**.** *There exists $f \in C^\infty(\mathbb{R}^3)$ such that the differential equation $Pu = (D_{x_1} + iD_{x_2} + 2i(x_1 + ix_2)D_{x_3})u = f$ does not have a distributional solution u in any neighborhood of 0. Here, $D_{x_j} = \partial_{x_j}/i$.*

Remark 1.1. One can show that this differential equation cannot be solved in any open set in $\mathbb{R}^3$.

Proof. This argument is due to Hörmander. Let $\Omega \subseteq \mathbb{R}^3$ be an open neighborhood of 0. What it means for $u \in D^1(\Omega)$ to solve this equation is that for all test functions $\varphi \in C_0^\infty(\Omega)$,

$$\underbrace{P_u(\varphi)}_{u(-P\varphi)} = f(\varphi) = \int f\varphi\,dx.$$

Therefore, for any compact set $K \subseteq \Omega$, there exist C, m such that

$$|f(\varphi)| \leq C \sum_{|\alpha| \leq m} \sup_K |\partial^\alpha (P\varphi)|$$

when $\varphi \in C_0^\infty(\Omega)$ with $\mathrm{supp}(\varphi) \subseteq K$.

Let $W = \{\varphi \in C_0^\infty(\Omega) : \mathrm{supp}(\varphi) \subseteq L\}$ with the locally convex topology given by the seminorms $\varphi \mapsto \sum_{|\alpha| \leq m} \sup |\partial^\alpha P\varphi|$ (only countable many seminorms occur). $F = C^\infty(\mathbb{R}^3)$, which is Fréchet. Now consider the bilinear map $B : E \times F \to \mathbb{C}$ given by $(\varphi, f) \mapsto \int f\varphi \, dx$. B is continuous in f for any fixed φ. B is also continuous in φ if the equation $Pu = f$ has a solution $u \in D^1(\Omega)$, in view of the above inequality.

We claim that the map B is not continuous provided that $0 \in \mathrm{int}(K)$. Assume that B is continuous. Then there exist a compact $L \subseteq \mathbb{R}^3$, C, and m such that

$$|f(\varphi)| \leq C \left(\sum_{|\alpha| \leq m} \sup |\partial^\alpha P\varphi| \right) \left(\sum_{|\alpha| \leq m} \sup_L |\partial^\alpha f| \right)$$

for all $\varphi \in C_0^\infty$ with $\mathrm{supp}(\varphi) \subseteq K$ and $f \in C^\infty(\mathbb{R}^3)$.

The idea is to show that the estimate is not valid by constructing a quasimode of P; we want to have φ such that $P\varphi \approx 0$ and $\varphi \approx 1$.[1] The form of P gives us that $P(x_1^2 + x_2^2 + ix_3) = 0$. Consider

$$w(x) = \frac{1}{i} \left[-x_1^2 - x_2^2 - ix_3 + (x_1^2 + x_2^2 + ix_3)^2 \right].$$

This satisfies $Pw = 0$. Note that $w = \frac{1}{i}\left[-|x|^2 - ix_3 + O(|x|^3)\right]$, so $\mathrm{Im}(w) = |x|^2 + O(|x|^3) \sim |x|^2$ near 0. Let $\chi \in C_0^\infty(\mathbb{R}^3)$ be such that $\chi = 1$ near 0 and such that $\mathrm{Im}(w) \geq |x^2|/2$ on $\mathrm{supp}(\chi)$. Let $V_\lambda(x) = \chi(x) e^{i\lambda w(x)} \in C_0^\infty$ with $\lambda \gg 1$. Then $\mathrm{supp}(v_\lambda) \subseteq K$, and $|v_\lambda| \sim e^{-\lambda|x|^2}$. Take $v_\lambda = \varphi$ in the inequality. Then $Pv_\lambda = (P\chi)e^{i\lambda w} = O(e^{-c\lambda})$ with $c > 0$. We get

$$\sum_{|\alpha| \leq m} \sup |\partial^\alpha P\varphi| = O(\lambda^m e^{-c\lambda}) \xrightarrow{\lambda \to \infty} 0.$$

Take $f(x) = f_\lambda(x) = e^{i\lambda x_3} \lambda^3 h(\lambda x)$ for $0 < h \in C_0^\infty$ with $\int h = 1$. The right hand side in the inequality is $O(\lambda^m e^{-c\lambda} \lambda^{3+M})$, which goes to 0 as $\lambda \to \infty$. The left hand side is

$$\int e^{i\lambda x_3} \lambda^3 h(\lambda x) \chi(x) e^{i\lambda w(x)} \, dx = \int e^{ix_3} h(x) \chi(x/\lambda) e^{i\lambda w(x/\lambda)} \, dx \xrightarrow{\lambda \to \infty} \int h = 1.$$

We get that the set of $f \in C^\infty$ such that the equation $pu = f$ has a solution $u \in D^1(\Omega)$ is of the first category. □

[1] Up to this point in the proof, we have not used the form of the operator P at all. This argument shows that if we can find a quasimode for any operator P with this property, then we can show that P has no solutions in this sense.